Measurement-Based Teleportation Along Quantum Spin Chains

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We consider teleportation of an arbitrary spin-$\frac{1}{2}$ target quantum state along the ground state of a quantum spin chain. We present a decomposition of the Hilbert space of the many-body quantum states into 4 vector spaces. Within each of these subspaces it is possible to take any superposition of states, and use projective measurements to perform unit fidelity teleportation. We also show that all total spin-0 many-body states belong in the same space, so it is possible to perform unit fidelity teleportation over any one-dimensional spin-0 many-body state. We generalise to n-Bell states, and present some general bounds on fidelity of teleportation given a general state of a quantum spin chain.

Quantum many-body spin Hamiltonians often have highly entangled ground states, and an understanding of entanglement may in turn lead to a greater understanding of the physics of many-body quantum systems with strong correlations[1]. However, a definitive measure of multipartite entanglement is yet to be found, though there exist several proposed measures [2, 3, 4]. In this Letter, we characterise the entanglement content of the ground states of several quantum spin chain Hamiltonians by its ability to teleport a state to an arbitrary site.

The protocol used for teleportation will be based purely upon measurement, local unitary operations, and classical communication. It has recently been noted that these operations are sufficient for universal QC (quantum computation)[5, 6, 7], particularly as manifest in cluster state QC. Hence, characterising the ability of a system to perform high fidelity teleportation is useful for the purposes of QC.

Teleportation over any Bell state by Bell basis measurements is well understood[8]. We begin extending this protocol by considering $\mathcal{L}$ pairs of particles in tensor products of Bell states, which provide a basis for the Hilbert space. We then show that we can decompose this space into classes of states which are equivalent for the purposes of teleportation. The ground states of several quantum spin chain Hamiltonians are considered. Finally, we introduce a two component quantity, whose magnitude lower bounds the average fidelity of a teleported state.

Tensored Bell basis teleportation. Let us denote our 2-dimensional spin space as V, with basis $|\uparrow\rangle, |\downarrow\rangle$. Let v^i represent the standard Bell basis states, where the v^0 is the singlet state,

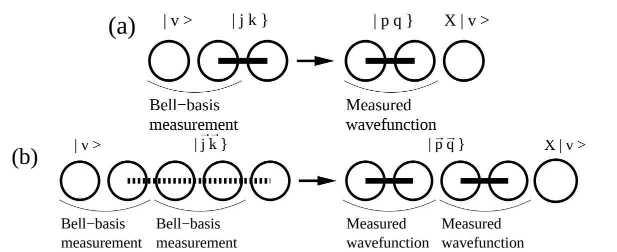

Figure 1: A schematic representation of the proposed teleportation protocol, showing input state $|v\rangle$ and final state $X|v\rangle$. X is known from the measurement outcome, (see Table I). (a) A Bell basis measurement over a single pair of maximally entangled spins, as in Eq. 2, represented by a solid line between two sites. (b) A schematic representation of our protocol for a general spin quantum state, as in Eq. 3, with non-trivial entanglement, represented by the dashed line. Here we assume that the state $\left|\vec{j}\vec{k}\right\}$ lies within a Bell subspace, yielding a pure output state, $X|v\rangle$.

$$v^0 = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right). \quad (1)$$

The other Bell states, v^i may be written as $v^i = \left(I \otimes X^i\right) v^0$, where $X^0 = \sigma_I$, $X^1 = \sigma_x$, $X^2 = -\sigma_z$, and $X^3 = i\sigma_y$, where the σ matrices are the usual Pauli matrices for spin-$\frac{1}{2}$ particles. These states are simultaneous eigenstates of $\sigma^x \otimes \sigma^x$ and $\sigma^z \otimes \sigma^z$. Let us introduce the notation $|kl\}$ for Bell states such that $\sigma^x \otimes \sigma^x |kl\} = k\,|kl\}$, and $\sigma^z \otimes \sigma^z |kl\} = l\,|kl\}$. Multipartite states which are amenable to use for teleportation turn out to be simultaneous eigenstates of similar $\mathcal{L}$-qubit operators.

Let $P^{|jk\}}$ denote the projections onto the state $|jk\}$. Given a target state which we wish to teleport, $|v\rangle$, and

$P^{\|pq\}}$ \ $\|jk\}$	$\|--\}$	$\|-+\}$	$\|+-\}$	$\|++\}$
$P^{\|--\}}$	$-X^0$	$-X^1$	$-X^2$	$-X^3$
$P^{\|-+\}}$	X^1	X^0	$-X^3$	$-X^2$
$P^{\|+-\}}$	X^2	X^3	X^0	X^1
$P^{\|++\}}$	$-X^3$	$-X^2$	X^1	X^0

Table I: Unitary correction, X^{jk}_{pq}, to be inverted on the final qubit, where we start with a quantum state $|v\rangle \otimes |jk\}$ and project onto $|pq\}$ with $P^{|pq\}} = |pq\}\{pq| \otimes I$. These correspond to either no correction (X^0), a bit flip (X^1), a phase flip (X^2), or both a bit and a phase flip (X^3)

a two particle Bell state $|jk\}$ over which we wish to teleport it, we can simply perform a Bell-basis measurement across the target state, and half of the two-particle state, as in Fig. 1 (a). We can make the following decomposition:

$$|v\rangle \otimes |jk\} = \frac{1}{2}\sum_{p,q} |pq\} \otimes X^{jk}_{pq}\,|v\rangle\,. \tag{2}$$

where X^{jk}_{pq} is one of the matrices X^i, as in Table I. We can identify X^{jk}_{pq} with a unitary correction, which we must invert to yield the target state $|v\rangle$. We now extend this standard result[8] to larger systems by making use of the fact that the matrices X^i form a group under composition.

For an even number $L = 2\mathcal{L}$ of spin-$\frac{1}{2}$ particles, we define the L-particle state $\left|\vec{j}\vec{k}\right\} = |j_1k_1\} \otimes \ldots \otimes |j_\mathcal{L}k_\mathcal{L}\}$, where $\vec{j} = (j_1, j_2, \ldots, j_\mathcal{L})$ is a $\mathcal{L}$-dimensional vector. We can then make the following decomposition

$$|v\rangle \otimes |\,\vec{j}\vec{k}\,\} = \frac{1}{2^\mathcal{L}}\sum_{\vec{p},\vec{q}} |\,\vec{p}\vec{q}\,\} \otimes X^{j_\mathcal{L}k_L}_{p_\mathcal{L}q_\mathcal{L}} \ldots X^{j_1k_1}_{p_1q_1}\,|v\rangle \tag{3}$$

Since the corrections X^i are closed under composition, we may write $X^{j_\mathcal{L}k_\mathcal{L}}_{p_\mathcal{L}q_\mathcal{L}} \ldots X^{j_1k_1}_{p_1q_1} = X^{\vec{j}\vec{k}}_{\vec{p}\vec{q}} = \pm X^i$ for some i. We may decompose the L-qubit Hilbert space V^L into 4 subspaces, each of which has a different total correction X^i. For any state within a given subspace $V^L_{[kl]}$, we are able to perform unit fidelity teleportation with any set of measurement results. This decomposition turns out to be independent of $\vec{p}$ and $\vec{q}$ [22]. Decomposing the Hilbert space as

$$V^L = V^L_{[--]} \oplus V^L_{[-+]} \oplus V^L_{[+-]} \oplus V^L_{[++]}, \tag{4}$$

the corrections are in 1-1 correspondence with the assignments given in Table I.

Physically, each component of the decomposition corresponds to all of the states which will give the same unitary correction after the series of Bell basis measurements. We refer to the 4 subspaces as Bell subspaces since they reduce to the Bell basis in the two-qubit case.

The decomposition of V^L into 4 subspaces with different corrections turns out to be extremely useful in understanding systems which have ground states lying entirely in one of these subspaces. We present several model Hamiltonians for which this is true. It is interesting to note that all of these ground states are similar to spin liquid states[9], since every member of $V^L_{[kl]}$ has every 1-qubit reduced density matrix equal to $\frac{1}{2}I$. Hence each site is also maximally entangled with the rest of the chain. Any state in one of these subspaces has maximum localisable entanglement with respect to Bell measurements[10].

Spin-$\frac{1}{2}$ Next-Neighbour Hamiltonians. A family of nearest- and next-nearest neighbour antiferromagnetic spin exchange Hamiltonians is parameterised, with $\beta > 0$ by

$$H_{n,nn} = \sum_{i=1}^{N} \hat{S}_i.\hat{S}_{i+1} + \beta\hat{S}_i.\hat{S}_{i+2} \tag{5}$$

for which $\beta = \frac{1}{2}$ yields the Majumdar-Ghosh Hamiltonian. The ground state of the Majumdar-Ghosh Hamiltonian[11] is simply comprised of a tensor product of singlets, $\otimes_{k=1}^{N/2}|v^0\rangle$, and one may use this to perform unit fidelity teleportation with repeated Bell basis measurements along the chain. However, it is possible to show that the ground state of Eq. 5 for any value of $\beta > 0$ lies within a Bell subspace, including the specific case of $\beta = 0$, which corresponds to the Heisenberg Hamiltonian.

More generally, it can be shown rigorously that any quantum state with a total spin-0 lies within one of the Bell subspaces, $V^L_{[kl]}$, which we outline as follows. Given $\Psi^0 = v^0 \otimes \ldots \otimes v^0$, it is clear that Ψ^0 must have total spin 0. Further, any permutation of sites in Ψ^0 must also have spin 0, and belong to the same subspace, $V^L_{[j^0k^0]}$[23]. In fact, it is possible to decompose every spin-0 state as a sum over permutations of the sites of Ψ^0[12]. This is known in the chemistry literature as resonant valence bond theory[13].

It is interesting to note that for the antiferromagnetic Ising Hamiltonian[14], there are two degenerate product form ground states, $|\Phi^-\rangle = |\uparrow\downarrow\ldots\downarrow\rangle$ and $|\Phi^+\rangle = |\downarrow\uparrow\ldots\uparrow\rangle$. The decomposition of these will have equal support in more than one Bell subspace.

$$\left|\Phi^\pm\right\rangle = (|+-\} \pm |--\}) \otimes \ldots \otimes (|+-\} \pm |--\})$$

However, the superposition, $|\Phi^0\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Phi^-\rangle)$ has support on only one Bell subspace. Further, $|\Phi^0\rangle$ has only one ebit[24] of entanglement - yet even so, we can use this state to perform unit fidelity teleportation over an arbitrary distance.

Spin-1 Models. The AKLT model[15] can be related to the antiferromagnetic Heisenberg model for spin-1, and is given by the specific case of $\alpha = \frac{1}{3}$ in the class of Hamiltonians

$$H_{AKLT} = \sum_{i=1}^{N} \hat{S}_i.\hat{S}_{i+1} + \alpha(\hat{S}_i.\hat{S}_{i+1})^2 \qquad (6)$$

We may decompose each spin-1 site, i, as two virtual spin-$\frac{1}{2}$ sites, $i, \bar{i}$, and project onto the spin-1 subspace. For an N site chain with spin-$\frac{1}{2}$ boundary conditions, we may write the ground state of the AKLT model as[16]

$$|\psi_{AKLT}\rangle = (\otimes_{k=1}^{N} A_{k\bar{k}})|I\rangle \qquad (7)$$

where $|I\rangle = \otimes_{k=0}^{N} |I_{\bar{k}k+1}\rangle$ is a product of singlets, and $A_{k\bar{k}}$ projects the spins at sites k and $\bar{k}$ onto their symmetric subspace. The operation of projecting out the singlet components leaves the state in the Bell subspace which also contains the singlet quantum states.

Verstraete *et al.*[16] show that the projection onto Bell states in the $i \otimes \bar{i}$ space is achievable with only single particle measurements. Hence, one can use only single particle spin-1 measurement to teleport a spin-$\frac{1}{2}$ state along a AKLT chain. Moreover, a linear spin-1 Heisenberg antiferromagnetic may exhibit spin-$\frac{1}{2}$ degrees of freedom at the boundaries, as evidenced by both numerical[17] and experimental results[18]. It may be possible to couple the target state into the spin chain by a Bell basis measurement over the target state and the boundary spin-$\frac{1}{2}$ degrees of freedom. Hence the entire teleportation problem may reduce to an initial Bell basis measurement and single particle measurements on spin-1 particles (see also [19]).

Non-Bell basis Measurements. Experimentally, it is very difficult to perform Bell basis measurements directly, and we now consider teleportation using only single particle measurements. To affect a Bell basis measurement, we may use the fact that there is a similarity transformation between Bell basis projection operators and a product of single particle projections, by using an entangling operation, U:

$$P^j = U^\dagger P_2^{j_2} P_1^{j_1} U \qquad (8)$$

where j_1 and j_2 are functions of j set by the unitary, U. Projections onto different sites commute, and hence, we may decompose our Bell measurement protocol as an application of entangling unitaries onto the whole quantum state, followed by a complete set of single particle projective measurements. The reduced requirements of single particle measurements and entangling operations are very similar to the requirements for cluster state teleportation[6].

Fidelity. Our protocol involves using a channel quantum state, $|\psi\rangle$, which we believe to belong to the subspace $V_{[pq]}^L$, performing Bell basis measurements along the chain, and then applying the appropriate correction. Let us decompose our channel state as

$$|\psi\rangle = \sum_{i,j=\pm} c_{i,j}|\phi_{i,j}\rangle \qquad (9)$$

where $|\phi_{i,j}\rangle$ are orthonormal vectors, each lying in subspaces $V_{[ij]}^L$. The condition that $|\psi\rangle$ lies within the subspace $V_{[pq]}^L$ implies $c_{i,j} = \delta_{i,p}\delta_{j,q}$, which is only approximately satisfied for real quantum states. Projecting into the Bell basis yields the measurement result $\left|\vec{a}\vec{b}\right\}$ with probability, $p_{|\vec{a}\vec{b}\}}$. If the target quantum state is $|v\rangle$, then the resulting quantum state will be $\left|\vec{a}\vec{b}\right\} \otimes |v'\rangle$ where $|v'_{\vec{a}\vec{b}}\rangle \propto \sum_{i,j=\pm} c_{i,j} X_{\vec{a}\vec{b}}^{ij}|v\rangle$ is a superposition of different corrections onto $|v\rangle$. The fidelity for a particular state $|v\rangle$ is given by $\mathcal{F} = \left|\langle v'_{\vec{a}\vec{b}}|X_{ab}^{pq}|v\rangle\right|^2$.

Teleportation Parameter. It is desirable to a define a quantity which we can use to characterise the fidelity of teleportation achievable by a measurement protocol over a quantum state. For this purpose, for any quantum spin state $|\Psi_0\rangle$, we define a quantity

$$\mathcal{O}(\Psi_0) = \sum_{\alpha=x,y,z} \left|\langle\Psi_0| \otimes_{k=1}^{L} \sigma_k^\alpha |\Psi_0\rangle\right| \qquad (10)$$

where $|\mathcal{O}| = 4\,|c_{p,q}|^2 - 1$.

We note that $\mathcal{O}(\Psi_0)$ is related to non-local correlations in $|\Psi_0\rangle$, bearing some similarity to a string order parameter[21]. We now show that $\mathcal{O}(\Psi_0)$ can be used to give a lower bound on the fidelity of teleportation through $|\Psi_0\rangle$. When $\mathcal{O}$ is close to 3, we satisfy the condition $c_{i,j} \simeq \delta_{i,p}\delta_{j,q}$, with equality when $\mathcal{O} = 3$.

The minimum possible fidelity given perfect Bell basis projections is a monotonically decreasing function of $\mathcal{O}$, as seen in Fig. 2. Note that the fidelity $\mathcal{F}$ is not uniquely defined by $\mathcal{O}$, also being dependent on $|v\rangle$ and the measurement history. Furthermore, for an arbitrary channel, measurement in the Bell basis is not necessarily optimal. These facts are reflected by the following inequality:

$$\mathcal{F} \geq \frac{\mathcal{O} - 1}{2} \qquad (11)$$

Generalisations. Generalisation to higher spin is possible, where a generalised N-Bell state is given by

$$|AB\} = \sum_{i=0}^{N-1} \omega^{Bi}|i(i+A)\rangle$$

where $s = \frac{N-1}{2}$ is the spin, $\omega = \sqrt[N]{1}$ is the primitive Nth root of 1, and $A, B \in \{-s, \ldots, s\}$ and $|ij\rangle$ are the

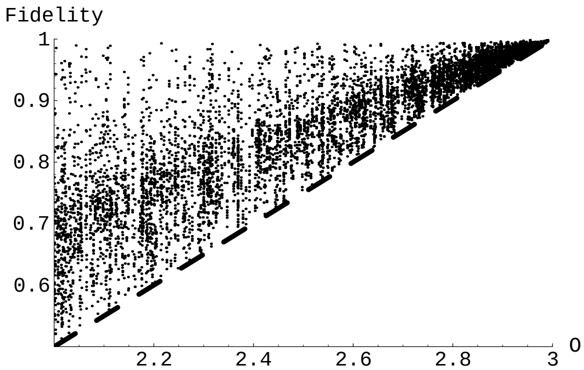

Figure 2: Data points correspond to the fidelity of teleportation for a range of target states. The fidelity is shown against the quantity $\mathcal{O}$ as defined in Eq. 10 for 2000 4-qubit channels. The dashed line is a lower bound for these points and is given by the inequality in Eq. 11. This clearly shows a tendency towards a fidelity, $\mathcal{F} = 1$ as $\mathcal{O}$ tends to 3.

usual two-particle spin-s basis states. In analogy to the interpretation of σ_x as a bit flip, and and σ_z as a phase flip, we introduce the permutation and phase correction matrices as:

$$P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \omega & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega^{N-1} \end{pmatrix}$$

As in the case of spin-$\frac{1}{2}$ particles, the corrections P^i and Q^i for $i = 1, 2, \dots, N$ form a group, and we may perform exactly the same procedure of generalised Bell-basis measurement, followed by a cumulative correction at the end of the procedure.

Cluster State QC. By inspection, the regular 2-site cluster state [6], $|\downarrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\uparrow\uparrow\rangle$ does not lie in any of the Bell subspaces, $V^L_{[i,j]}$, since it is a linear superposition of two Bell basis states. However, we may rewrite this state as $|\downarrow\rightarrow\rangle + |\uparrow\leftarrow\rangle$, where $|\rightarrow\rangle = |\downarrow\rangle + |\uparrow\rangle$ and $|\leftarrow\rangle = |\downarrow\rangle - |\uparrow\rangle$ are the usual eigenvectors of σ_x. This state now resembles a Bell state, and we may repeat our entire construction by using the z, x basis rather than the z, z basis. More generally it can be shown that the 1-dimensional L-qubit cluster states of [6] can be mapped to the subspace $V^L_{[++]}$ through a sequence of local unitary transformations. This may be expected, as our main result strongly resembles cluster state methods in the use of local measurement and feed forward. The main difference is in the initialisation of the cluster state, in which entanglement is provided by a series of controlled-phase entangling gates, in contrast to our requirement that the spin system is in a Bell subspace.

We have shown that by using a Bell-basis measurement protocol, it is possible to decompose the Hilbert space into Bell subspaces, using a decomposition into independent corrections. Further, we have shown that if any quantum state belongs to a Bell subspace, then we may perform unit fidelity teleportation with it, using only measurements in the generalised Bell basis. We have presented several model Hamiltonians for which the ground state is amenable to our teleportation protocol. Some alternative measurement schemes have been presented, including a scheme for a spin-1 system. Finally, we have presented a complex number valued parameter whose magnitude provides a lower bound on the fidelity of teleportation using our measurement protocol. Finally, we note that we have only considered the ability to teleport one spin quantum state to an arbitrary site[25] within a spin chain. Thus, v^0 and $v^0 \otimes v^0$ both have $\mathcal{O} = 3$, despite having a different number of ebits. There may be further insights to be gained into the entanglement content of a system, by considering the ability to teleport several spin states to arbitrary sites.

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[22] For any state $|jk\rangle$, we may change the projection from P^{pq} to P^{rs}, and the total unitary correction will change by $X^{jk}_{pq}(X^{jk}_{rs})^{-1}$, which is independent of $\{j,k\}$.
[23] More formally the spaces, $V^L_{[jk]}$, are invariant under an action of the symmetric group
[24] 1 ebit is equivalent to one distillable singlet.
[25] We have not assumed any special ordering on the sites or the measurements.